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A NEW GENERALIZATION OF THE EXPONENTIAL-POISSON DISTRIBUTION USING ORDER STATISTICS

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ABSTRACT

This paper introduces a new family of lifetime distributions, using the ascendant order statistics. The proposed distribution is called the exponential-generalized truncated Poisson (EGTP) distribution. Our approach follows the same procedure as Adamidis and Loukas (1998) and generalizes the exponential Poisson distribution introduced by Kus (2007). We give general forms of the probability density function (pdf), the cumulative distribution (cdf), the reliability and failure rate functions of any order statistics. The parameters' estimation is attained by the maximum likelihood (ML) and the expectation maximization (EM) algorithms. The applied study is illustrated based on real datasets.

KEYWORDS: Order Statistics; Exponential Distribution; Failure Rate; Survivor Function; Truncated Poisson Distribution; Lifetime Distributions; EM Algorithm

Mathematics Subject Classification: 62E15, 62F10

INTRODUCTION

Lifetime distributions have been introduced, for modelling and analyzing real lifetime data in many areas of studies, such as engineering, computer science, biology, actuarial science, biomedical studies and reliability. The exponential and Weibull are the most common distributions used in lifetime and reliability analysis, in various fields of applied statistics. They give good insight into the nature and importance of the reliability and failure rate functions. The exponential distribution is often used in reliability theory, assuming constant failure rate (Balakrishnan and Basu, 1995; Barlow and Proschan, 1975). In recent years, a growing number of scholarly papers has been devoted to accommodate data where the underlying hazard rates present monotone shapes. Several lifetime models are proposed as extensions of the exponential distribution using a mixture of discrete and continuous distributions. For example, Adamidis and Loukas (1998) proposed the exponential-geometric (EG) distribution; with decreasing failure rate (see also, Adamidis et al., 2005 and Silva et al., 2010). In the same way, Kus (2007) introduced the exponential-Poisson (EP) distribution that is generalized by Hemmati et al. (2011), using an exponential-Weibull (EW) distribution. Tahmasbi and Rezaei (2008, 2008a), proposed the exponential-logarithmic (EL) distribution. Chahkandi and Ganjali (2009), proposed the exponential power-series (EPS) distributions. This family includes the compound exponential-binomial (EB) distribution. Barreto-Souza and Cribari-Neto (2009), generalized the EP distribution by exponentiation. Barreto-Souza et al. (2011), extended the EG distribution, using the mixture of the geometric and Weibull distributions. Morais and Barreto-Souza (2011), introduced the Weibull power-series (WPS) distributions that extend the EPS distribution. The geometric-exponential Poisson (GEP) distribution is proposed by Nadarajah et al (2013). Barreto-Souzaa and Bakouch (2013) introduced another distribution, by mixing the exponential with the Poisson–Lindley distribution. All these lifetime distributions previously carried out are with decreasing or increasing failure rates (DFR or IFR). Indeed, these distributions come from the idea of modelling a system reliability based on the reliability of its components (parallel, series and combined system). Thus, previous papers are focused only on the study of the minimum or the maximum lifetime, i.e. the first or the last order statistics.

In this paper, we generalize these distributions modelling the time of the first (or the last) failure, to a distribution more appropriate for modelling any k^{th} order statistic (second, third, or any k^{th} lifetime). For example, one may let $X_{(1)} < X_{(2)} < \cdots < X_{(Z)}$ be the order statistics of Z independent observations, of time-periods $T = (T_1, T_2, \ldots, T_z)$, previous studies may be focused on the minimum, $X_{(1)} = \min\{T_i\}_{i=1}^{Z}$, or the maximum, $X_{(Z)} = \min\{T_i\}_{i=1}^{Z}$, lifetime. We may be interested in the k^{th} duration and then determine the probability density function (pdf) for k^{th} order statistic. The random number Z may be modeled, using the Poisson distribution. The lifetime is an exponentially distributed random variable. One may also determine the distribution of the k^{th} smallest value of failure time. Then, the k^{th} order statistic is the lifetime of a $(n-k+1)$ -out-of-n system. We propose a new family of lifetime distributions by mixing the exponential and the generalized truncated Poisson distributions, called the exponential-generalized truncated Poisson (EGTP) distribution. We show that the minimum lifetime (Kus, 2007) is a special case of our EGTP distribution.

The paper is organized as follows: Section 2 presents the proposed EGTP distribution and the pdf, for some special cases. The moment-generating function (mgf) and the rth moment are presented in section 3. The reliability and failure rate functions are discussed in section 4. Random number generation is shown in section 5. The parameters' estimation are discussed in section 6 and the estimates of the parameters are obtained by the maximum likelihood (ML) and the expectation maximization (EM) algorithms. Finally, the application study is illustrated in the last section.

The Proposed Distribution

Let $T = (T_1, T_2, \ldots, T_z)$ be iid exponential random variables with the pdf $f(t) = \theta e^{-\theta t}$, for $t \geq 0$ and $\theta > 0$. Z is the random number of unit in a system, that follows the truncated Poisson distribution, with a probability function $P_l(Z = z)$ defined in equation (1):

$$P_l(Z = z) = \frac{e^{-\lambda}\lambda^z}{\Gamma(z+1)[1-\sum_{i=0}^{l} P(Z=i)]} \; ; \; z = l+1, l+2, \ldots \tag{1}$$

where $\lambda > 0$, $\Gamma(z+1) = z!$ and $P(Z = i)$ is a Poisson distribution with pdf:

$$P(Z = i) = \frac{e^{-\lambda}\lambda^i}{\Gamma(i+1)} \; ; \; \lambda > 0 \; and \; i = 0, 1, 2, \ldots.$$

Let $X_{(k)}$ be the k^{th}-smallest value of lifetime (the k^{th} order statistic). Then, its pdf is given by (David, 1981, p. 9; Balakrishnan and Cohen, 1991, p. 12):

$$f_k(x|z,\theta) = \frac{\theta\Gamma(z+1)}{\Gamma(k)\Gamma(z-k+1)} e^{-\theta(z-k+1)x}(1-e^{-\theta x})^{k-1} \qquad \theta, x > 0 \tag{2}$$

The joint probability density is derived from equations (1) and (2) as:

$$g_k(x,z|\lambda,\theta)=\frac{\theta\lambda^z e^{-\theta(z-k+1)x-\lambda}(1-e^{-\theta x})^{k-1}}{\Gamma(k)\Gamma(z-k+1)\left[1-\sum_{i=0}^{k-1}P(Z=i)\right]} \quad ; \; k=1,2,\dots,z \tag{3}$$

where, X and Z are the lifetime of a system and the last order statistic, respectively. Equation (3) is derived for the ascending order $X_{(1)} < X_{(2)} < \dots < X_{(Z)}$. This joint probability density function is determined by compounding a truncated at $l = k-1$ Poisson distribution and the density of the k^{th} order statistic $f_k(x|z,\theta)$ for $(k = 1,2,\dots,z)$. The use of the truncated at $k-1$ Poisson distribution is motivated by mathematical interest because we are interested in the k^{th} order statistic. There is a left-truncation scheme, where only $(z-k+1)$ individuals (or units) who survive a sufficient time are included, i.e. we observe only individuals or units with $X_{(k)}$ exceeds the time of the event that truncates individuals. In comparison with the formulation of Kus (2007) we consider the k^{th}-smallest value of lifetime instead of the minimum lifetime $X_{(1)} = \min\{T_i\}_{i=1}^{Z}$. Thus, our proposed new family of lifetime distributions, named the exponential-generalized truncated Poisson (EGTP) distribution, is the marginal density distribution of x given by:

$$g(x|\lambda,\theta,k)=\frac{\theta\lambda^k e^{-\lambda(1-e^{-\theta x})-\theta x}(1-e^{-\theta x})^{k-1}}{\Gamma(k)\left[1-\sum_{i=0}^{k-1}P(Z=i)\right]} \quad ; \; k=1,2,\dots,z \tag{4}$$

Our distribution is more appropriate for modelling any k^{th} order statistic (2^{nd}, 3^{rd} or any k^{th} lifetime). We show later that the minimum lifetime (Kus, 2007) is a special case the EGTP distribution.

Since, $\Gamma(k)\left[1-\sum_{i=0}^{k-1}P(Z=i)\right]=IG(\lambda,k)$

where IG is the incomplete Gamma function defined by:

$$IG(u,s)=\int_0^u t^{s-1}e^{-t}dt$$

Then, the final form of the pdf of x is given by:

$$g(x|\lambda,\theta,k)=\frac{\theta\lambda^k e^{-\lambda(1-e^{-\theta x})-\theta x}(1-e^{-\theta x})^{k-1}}{IG(\lambda,k)} \quad ; \; k=1,2,\dots,z \tag{5}$$

where, $\lambda>0$ is the "shape" parameter and $\theta>0$ is the "scale" parameter. Also, the cumulative distribution function of x corresponding to the pdf in equation (5) is given by:

$$G(x|\lambda,\theta,k)=\frac{IG[y(x),k]}{IG(\lambda,k)} \quad ; \; k=1,2,\dots,z \tag{6}$$

where $y(x)=\lambda(1-e^{-\theta x})$

In table (1), we present the pdf in equation (5) for some special cases at the first, second and third order statistics. Table (1) shows that the particular case of the EGTP function, for $k = 1$, is the lifetime EP distribution due to Kus (2007). The pdf decreases strictly in x and tends to zero as $x \to \infty$. Note that, for $k = 1$ the EGTP distribution is strictly decreasing with a modal value equals to $\lambda\theta(1-e^{-\lambda})^{-1}$ given at $x = 0$. As $\lambda \to 0$ and $k = 1$, the EGTP distribution tends to an exponential with parameter θ.

Table (1): The Pdf for Some Special Cases

Order Statistics	k	pdf
First	$k = 1$	$\dfrac{\theta\lambda e^{-\lambda(1-e^{-\theta x})-\theta x}}{1-e^{-\lambda}}$
Second	$k = 2$	$\dfrac{\theta\lambda^2 e^{-\lambda(1-e^{-\theta x})-\theta x}(1-e^{-\theta x})}{1-e^{-\lambda}(\lambda+1)}$
Third	$k = 3$	$\dfrac{\theta\lambda^3 e^{-\lambda(1-e^{-\theta x})-\theta x}(1-e^{-\theta x})^2}{2-e^{-\lambda}(2\lambda+3)}$

Moment Generating Function and r^{th} Moment

Suppose x has the pdf in equation (5), then moment generating function (mgf) is given by:

$$E(e^{tx}) = \frac{\Gamma(r+1)\lambda^k}{IG(\lambda,k)} \sum_{i=0}^{\infty}\sum_{j=0}^{i+k-1} \frac{(-1)^{i+j}\lambda^i \Gamma(i+k)(j+1-\frac{t}{\theta})^{-1}}{\Gamma(i+1)\Gamma(j+1)\Gamma(i+k-j)} \quad ; \quad k = 1, 2, \ldots, z$$

and hence the r^{th} the moment is given by:

$$E(X^r) = \frac{\lambda^k \Gamma(r+1)}{\theta^r IG(\lambda,k)} \sum_{i=0}^{\infty}\sum_{j=0}^{i+k-1} \frac{(-1)^{i+j}\lambda^i \Gamma(i+k)(j+1)^{-r-1}}{\Gamma(i+1)\Gamma(j+1)\Gamma(i+k-j)} \quad ; \; k = 1, 2, \ldots, z$$

Reliability and Failure Rate Functions

The reliability function $S(x) = Pr\{X \geq x\} = 1 - G(x) = \int_x^{\infty} f(t)dt$ is given by:

$$S(x|\lambda,\theta,k) = 1 - \frac{IG[y(x),k]}{IG(\lambda,k)} \quad ; \quad k = 1, 2, \ldots, z \tag{7}$$

In the literature on reliability theory, one may see Barlow and Proschan (1975, 1981) and Basu (1988). The failure rate function $h(x)$ is the "the rate of event occurrence per unit of time". We define a failure rate function as in the Barlow and Proschan (1965) by $h(x) = g(x)/S(x)$:

$$h(x|\lambda,\theta,k) = \frac{\theta\lambda^k e^{-\lambda(1-e^{-\theta x})-\theta x}(1-e^{-\theta x})^{k-1}}{IG(\lambda,k) - IG[y(x),k]} \; ; \; k = 1, 2, \ldots, z \tag{8}$$

The hazard function is analytically related to the time-failure probability distribution. It leads to the examination of increasing (IFR) or decreasing failure rate (DFR) properties of life-length distributions. G is an IFR distribution, if $h(x)$ increases for all X such that $G(X) < 1$.

In table (2) we present, the reliability function in equation (7) and the failure rate function in equation (8) for some special cases at the first, second and third order statistics. Note that we have 2 cases: $k = 1$ and $k \neq 1$. If $k = 1$, the hazard rate function is decreasing following kus (2007). In fact, if $x \to 0$ then $h(x\backslash\lambda, \theta, k) = \theta\lambda(1 - e^{-\lambda}) > 0$ and if $x \to \infty$ then $h(x\backslash\lambda, \theta, k) \to 0$.

If $k \neq 1$, there is an increasing hazard rate. Indeed, if $x \to 0$ then $h(x\backslash\lambda, \theta, k) \to 0$. If $x \to \infty$ then $h(x\backslash\lambda, \theta, k) > 0$.

Table 2: The Reliability and Failure Rate Functions for Some Special Cases

		Reliability
First Order	k=1	$\frac{e^{\lambda e^{-\theta x}}-1}{e^{\lambda}-1}$
Second Order	k=2	$\frac{e^{\lambda e^{-\theta x}}(1+\lambda-\lambda e^{-\theta x})-1-\lambda}{e^{\lambda}-1-\lambda}$
Third Order	k=3	$\frac{e^{\lambda e^{-\theta x}}[2(1+\lambda-\lambda e^{-\theta x})+\lambda^2(1-e^{-\theta x})^2]-(2+2\lambda+\lambda^2)}{2e^{\lambda}-(2+2\lambda+\lambda^2)}$
		Failure Rate
First Order	k=1	$\frac{\theta\lambda e^{-\lambda(1-e^{-\theta x})-\theta x}}{e^{-\lambda}(e^{\lambda e^{-\theta x}}-1)}$
Second Order	k=2	$\frac{\theta\lambda^2 e^{-\lambda(1-e^{-\theta x})-\theta x}(1-e^{-\theta x})}{1+e^{\lambda e^{-\theta x}}-(\lambda+1)e^{-\lambda}}$
Third Order	k=3	$\frac{\theta\lambda^3 e^{-\theta x}(1-e^{-\theta x})^2}{\lambda^2(1-e^{-\theta x})^2+2\lambda(1-e^{-\theta x})+2-e^{-\lambda e^{-\theta x}}(\lambda^2+2\lambda+2)}$

Random Number Generation

The cdf of X in equation (6), $G(x\backslash\lambda, \theta, k)$, is a right-truncated Gamma. We can generate the random variable X using the following steps:

- Generate a random variable U defined on the interval $[0, \lambda]$ from the truncated Gamma distribution, following the algorithm proposed by Philippe (1997).
- Solve the nonlinear equation in y:

$$U = \frac{IG(y,k)}{IG(\lambda,k)}$$

- Calculate the values of X as:

$$X = -\frac{1}{\theta} ln\left(1 - \frac{y}{\lambda}\right)$$

For the case of the first order statistics, $k = 1$, we can generate X directly as the following:

$$X = -\frac{1}{\theta} ln\left(\frac{1}{\lambda} ln[(1-U)e^{\lambda} + U]\right)$$

where, U is a random variable with standard Uniform distribution.

Estimation

In this section, we will determine the estimates of the two-parameters λ and θ for our EGTP new family of distributions. Let $(X_1, X_2, \dots, X_n)$ be a random sample with observed values $(x_1, x_2, \dots, x_n)$ from the EGTP distribution with pdf in equation (5). The log–likelihood function given the observed values, $x_{obs} = (x_1, \dots, x_n)$, is:

$$\ell(\lambda,\theta / x_{obs},k) = -n\ln IG(\lambda,k) + n\ln(\theta) + nk\ln(\lambda) - \lambda\sum_{i=1}^{n}(1-e^{-\theta x_i}) - \theta\sum_{i=1}^{n}x_i$$
$$+(k-1)\sum_{i=1}^{n}\ln(1-e^{-\theta x_i}) \qquad ; \qquad k = 1,2,\dots,z$$

The associated gradients are:

$$\frac{\partial}{\partial\lambda}\ell(\lambda,\theta / x,k) = \frac{-n\lambda^{k-1}e^{-\lambda}}{IG(\lambda,k)} + \frac{nk}{\lambda} - \sum_{i=1}^{n}(1-e^{-\theta x_i}) \quad ; \quad k = 1,2,\dots,z$$

$$\frac{\partial}{\partial\theta}\ell(\lambda,\theta / x,k) = \frac{n}{\theta} - \lambda\sum_{i=1}^{n}x_i e^{-\theta x_i} - \sum_{i=1}^{n}x_i + (k-1)\sum_{i=1}^{n}\frac{x_i e^{-\theta x_i}}{1-e^{-\theta x_i}} \quad ; \; k = 1,2,\dots,z$$

The estimated EGTP parameters $\hat{\lambda}$ and $\widehat{\theta}$ can be found using the EM algorithm that handles the incomplete data problem (McLachlan and Krishnan, 1997; Dempster et al., 1977). This iterative procedure consists on repeatedly replacing the missing values with the new estimated ones to update the parameter estimates. The standard method used for MLEs is the Newton–Raphson algorithm which is also required for the M-step of the EM algorithm. The use of the algorithm needs second derivatives of the log-likelihood for all iterations. However, when the amount of the missed values (or information) in the data is relatively large, the EM method converges slowly than the Newton–Raphson algorithm (Little and Rubin, 1983). Recently, the EM algorithm has been used in several research papers such as in Adamidis et al. (2005), Karlis (2003), Ng et al. (2002), Adamidis (1999), Adamidis and Loukas (1998) and others.

To start the algorithm, we should define a hypothetical distribution of complete-data with pdf in equation (3) and then drive the conditional mass function as:

$$p_k(z\,|\,x,\lambda,\theta) = \frac{\lambda^{z-k}e^{-\theta(z-k+1)x-\lambda e^{-\theta x}}}{\Gamma(z-k+1)} \qquad ; \qquad k = 1,2,\dots,z$$

and,

E-step:

$$E_k(Z\,|\,x,\lambda,\theta) = k + \lambda e^{-\theta x} \qquad ; \qquad k = 1,2,\dots,z$$

M-step:

$$\lambda^{(r+1)} = \frac{\sum_{i=1}^{n}(k+\lambda^{(r)}e^{-\theta^{(r)}x_i})}{n\sum_{i=1}^{n}(1+\frac{(\lambda^{(r+1)})^{k-1}e^{-\lambda^{(r+1)}}}{IG(\lambda^{(r+1)},k)})} \qquad ; \quad k = 1,2,\dots,z$$

$$\theta^{(r+1)} = \frac{n}{\sum_{i=1}^{n} x_i \left(k + \lambda^{(r)} e^{-\theta^{(r)} x_i} - \frac{k-1}{1 - e^{-\theta^{(r+1)} x_i}}\right)} \quad ; \quad k = 1, 2, \ldots, z$$

Application Example

In the example application, we fit the EGTP new distribution to a real dataset and we compare it with the EB, EL, EP, EG, EPL, Gamma and Weibull distributions. The data set is from Kus (2007) and is analyzed by Barreto-Souza and Bakouch (2013), Chahkandi and Ganjali (2009) and Tahmasbia and Rezaeib (2008) to fit lifetime distributions with decreasing failure rate. The dataset represents 24 observations of "time intervals (in days) between successive earthquakes in the last century in North Anatolia fault zone" (Table 3):

Table 3: "Time Intervals of the Successive Earthquakes" (Kus 2007)

1163	3258	323	159	756	409	501	616
398	67	896	8592	2039	217	9	633
461	1821	4863	143	182	2117	3709	979

Table 4 shows the fitted parameters, calculated values of Kolmogorov–Smirnov (K-S) and their respective p-values. We compare some special cases of the EGTP distribution at 5% significance level. The K-S test shows that the EGTP distribution is an attractive alternative to the other models and it generalizes them to any k^{th} order statistics. Indeed, for k=1, The EGTP estimates are very similar to those obtained from the PE model. The new lifetime model provides a good fit to the dataset. The calculated K-S statistic is smaller than that obtained from the PE and its associated p-value is larger.

Table 4: The Goodness of Fit for Some Special Cases, For the First Dataset

Distributions	Estimates		K-S value	p-value
	$\hat{\theta}$	$\hat{\lambda}$		
EGTP:				
First order ($k = 1$)	3.61×10^{-4}	2.6170	0.0950	0.9820
Second order ($k = 2$)	5.56×10^{-4}	4.5600	0.1480	0.6680
Third order ($k = 3$)	7.31×10^{-4}	6.1520	0.1830	0.3980
Fourth order ($k = 4$)	8.84×10^{-4}	7.6420	0.2010	0.2880
EB	3.70×10^{-4}	0.1046	0.0985	0.9738
EL	4.14×10^{-4}	0.1260	0.0885	0.9885
EP	3.6×10^{-4}	2.6377	0.0972	0.9772
EPL	3×10^{-4}	0.5312	0.0712	0.9990
EG	3.3×10^{-4}	0.7369	0.0964	0.9690
Gamma	5×10^{-4}	0.7117	0.1235	0.8328
Weibull	8.12×10^{-4}	0.7854	0.1004	0.9690

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